

# THE BOTTOM OF THE SPECTRUM OF TIME-CHANGED PROCESSES AND THE MAXIMUM PRINCIPLE OF SCHRÖDINGER OPERATORS

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ABSTRACT. We give a necessary and sufficient condition for the maximum principle of Schrödinger operators in terms of the bottom of the spectrum of time-changed processes. As a corollary, we obtain a sufficient condition for the Liouville property of Schrödinger operators.

## 1. INTRODUCTION

In [12], we define the subcriticality, criticality and supercriticality for Schrödinger forms and characterize these properties in terms of the bottom of the spectrum of time changed processes. In the process, we prove the existence of a harmonic function (or ground state) of the Schrödinger form and study its properties. In particular, we show that it has a bounded, positive, continuous version which is invariant with respect to its Schrödinger semigroup. In this paper, we will show, as an application of this fact, the maximum principle and Liouville property of Schrödinger operators.

Let  $X$  be a locally compact separable metric space and  $m$  a positive Radon measure on  $X$  with full topological support. Denote by  $X_\Delta := X \cup \{\Delta\}$  the one-point compactification of  $X$ . Let  $\mathbf{M} = (\mathbf{P}_x, X_t, \zeta)$  be an  $m$ -symmetric Hunt process with lifetime  $\zeta = \inf\{t > 0 \mid X_t = \Delta\}$ . We assume that  $\mathbf{M}$  is irreducible and strong Feller. Let  $\mu = \mu^+ - \mu^-$  be a signed Radon smooth measure such that the positive (resp. negative) part  $\mu^+$  (resp.  $\mu^-$ ) belongs to the local Kato class (resp. the Kato class). We denote by  $A_t^{\mu^+}$  (resp.  $A_t^{\mu^-}$ ) the positive continuous additive functional in the Revuz correspondence to  $\mu^+$  (resp.  $\mu^-$ ). Put  $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$  and define the Feynman-Kac semigroup  $\{p_t^\mu\}_{t \geq 0}$  by

$$p_t^\mu f(x) = \mathbf{E}_x \left( e^{-A_t^\mu} f(X_t) \right).$$

We denote by  $\mathbf{M}^{\mu^+} = (P_x^{\mu^+}, X_t, \zeta)$  the subprocess of  $\mathbf{M}$  by the multiplicative functional  $\exp(-A_t^{\mu^+})$  and by  $(\mathcal{E}^{\mu^+}, \mathcal{D}(\mathcal{E}^{\mu^+}))$  the Dirichlet form generated by  $\mathbf{M}^{\mu^+}$ . Suppose that the negative part  $\mu^-$  is non-trivial and Green-tight with respect to

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$\mathbf{M}^{\mu+}$  (Definition 2.2 (2)). We then define  $\lambda(\mu)$  by

$$(1) \quad \lambda(\mu) := \inf \left\{ \mathcal{E}(u, u) + \int_X u^2 d\mu^+ \mid u \in \mathcal{D}(\mathcal{E}), \int_X u^2 d\mu^- = 1 \right\}.$$

$\lambda(\mu)$  is regarded as the bottom of the spectrum of the time-changed process of  $\mathbf{M}^{\mu+}$  by the continuous additive functional  $A_t^{\mu+}$ . We show in [11, Theorem 2.1] that the minimizer of (1) exists in the extended Dirichlet space  $\mathcal{D}_e(\mathcal{E}^{\mu+})$  and it can be taken to be strictly positive on  $X$ . The objective of this paper is to prove the maximum principle of Schrödinger forms by using the existence of the minimizer of (1). More precisely, let

$$(2) \quad \mathcal{H}^{ba}(\mu) = \{h \in \mathcal{B}(X) \mid h \text{ is bounded above, } p_t^\mu h \geq h\},$$

where  $\mathcal{B}(X)$  is the set of Borel functions on  $X$ . We here define the maximum principle by

(MP) If  $h \in \mathcal{H}^{ba}(\mu)$ , then  $h(x) \leq 0$  for all  $x \in X$ .

We will prove in Theorem 3.1 that under Assumption

$$(A) \quad \mathbf{E}_x \left( e^{-A_\infty^{\mu+}}; \zeta = \infty \right) = 0,$$

(MP) is equivalent to  $\lambda(\mu) > 1$ . For the proof of this, it is crucial that if  $\lambda(\mu) = 1$ , then the minimizer  $h$  in (1) has a bounded continuous version with  $p_t^\mu$ -invariance, i.e.,  $\mathbf{E}_x(\exp(-A_t^\mu)h(X_t)) = h(x)$  ([12, Lemma 5.16, Corollary 5.17]).

Let us introduce the space  $\mathcal{H}^b(\mu)$  of bounded  $p_t^\mu$ -invariant functions:

$$\mathcal{H}^b(\mu) = \{h \in \mathcal{B}_b(X) \mid p_t^\mu h = h\}.$$

We here define the *Liouville property* by

(L) If  $h \in \mathcal{H}^b(\mu)$ , then  $h(x) = 0$  for all  $x \in X$ .

We will show in Corollary 4.1 that under Assumption (A),  $\lambda(\mu) > 1$  implies (L).

We remark that Theorem 3.1 and Corollary 4.1 can be applied to non-local Dirichlet forms. In a remaining part of introduction, we treat these two properties for strongly local Dirichlet forms, which are regarded as an extension of symmetric elliptic operators of second order. In Berestycki-Nirenberg-Varadhan [2], they define a maximum principle for a uniformly elliptic operator of second order,  $L = M + c = a_{i,j}\partial_i\partial_j + b_i\partial_i + c$ , on a general bounded domain  $D$  of  $\mathbb{R}^d$ . Let  $u_0$  be the solution to the equation  $Mu = -1$  vanishing on  $\partial D$  in a suitable sense: define  $\mathcal{S}$  by the set of sequences  $\{x_n\}_{n=1}^\infty \subset D$  such that  $x_n$  converges to a point of the boundary  $\partial D$  and  $u_0(x_n)$  converges to 0. They say that the *refined maximum principle* holds for  $L$ , if a function  $h$  bounded above satisfies  $Lh \geq 0$  on  $D$  and  $\limsup_{n \rightarrow \infty} h(x_n) \leq 0$  for any  $\{x_n\}_{n=1}^\infty \in \mathcal{S}$ , then  $h \leq 0$  on  $D$ , and prove that  $L$  satisfies the refined maximum principle if and only if the principal eigenvalue  $\lambda_0$  of  $-L$  is positive.

Note that  $u_0$  equals  $\mathbf{E}_x(\tau_D)$ , where  $\mathbf{P}_x$  is the diffusion process with generator  $M$  and  $\tau_D$  is the first exit time from  $D$ . We see that if  $D$  is bounded (more generally, Green-bounded, i.e.,  $\sup_{x \in D} \mathbf{E}_x(\tau_D) < \infty$ ), then  $\mathcal{S}$  is identical to the set

of sequences  $\{x_n\}$  such that  $x_n \rightarrow \partial D$  and  $\mathbf{E}_{x_n}(\exp(-\tau_D)) \rightarrow 1$  as  $n \rightarrow \infty$  (Lemma 3.1, Remark 3.1). Considering this fact, we define

$$(3) \quad \mathcal{S} = \left\{ \{x_n\}_{n=1}^\infty \subset X \mid x_n \rightarrow \Delta \text{ and } \mathbf{E}_{x_n}(e^{-\zeta}) \rightarrow 1 \text{ as } n \rightarrow \infty \right\}.$$

Assume that  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local and set

$$\tilde{\mathcal{H}}^{ba}(\mu) = \left\{ h \mid \begin{array}{l} h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C(X) \text{ is bounded above, } \mathcal{E}^\mu(h, \varphi) \leq 0 \text{ for} \\ \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(X) \text{ and } \limsup_{n \rightarrow \infty} h(x_n) \leq 0 \text{ for } \forall \{x_n\} \in \mathcal{S} \end{array} \right\},$$

where  $C_0^+(X)$  is the set of non-negative continuous functions with compact support. Following [2], we here define the *refined maximum principle* by

**(RMP)** If  $h \in \tilde{\mathcal{H}}^{ba}(\mu)$ , then  $h(x) \leq 0$  for all  $x \in X$ .

We will show that  $\tilde{\mathcal{H}}^{ba}(\mu) \subset \mathcal{H}^{ba}(\mu)$  (Lemma 3.4), and thus see, as a corollary of Theorem of 3.1, that  $\lambda(\mu) > 1$  implies **(RMP)** (Theorem 3.2). We would like to emphasize that if  $D$  is bounded and  $L$  is symmetric, the principal eigenvalue  $\lambda_0$  of  $-L$  is positive if and only if  $\lambda(\mu) > 1$ . However,  $\lambda(\mu) > 1$  does not always imply  $\lambda_0 > 0$  for an unbounded domain  $D$ , while  $\lambda_0 > 0$  implies  $\lambda(\mu) > 1$  in general (Lemma 3.5).

When  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local, we set

$$\tilde{\mathcal{H}}^b(\mu) = \{h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C_b(X) \mid \mathcal{E}^\mu(h, \varphi) = 0, \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(X)\}$$

and define the property  $(\tilde{\mathbf{L}})$  by

$(\tilde{\mathbf{L}})$  If  $h \in \tilde{\mathcal{H}}^b(\mu)$ , then  $h(x) = 0$  for all  $x \in X$ .

We then see that if  $\mathbf{M}$  is conservative and **(A)** is fulfilled, that is,  $\mathbf{E}_x(\exp(-A_\infty^\mu)) = 0$ , then  $\tilde{\mathcal{H}}^b(\mu) \subset \mathcal{H}^b(\mu)$ , and consequently  $\lambda(\mu) > 1$  implies  $(\tilde{\mathbf{L}})$  (Corollary 4.2). Grigor'yan and Hansen [7] calls a measure  $\mu^+$  *big* if it satisfies **(A)**, and they prove that for the transient Brownian motion  $\mathbf{M} = (\mathbf{P}_x, B_t)$  on  $\mathbb{R}^d$ , if  $\mu^- \equiv 0$  and  $\mu^+$  is big, then  $(\tilde{\mathbf{L}})$  holds. Corollary 4.1 tells us that if  $\mu^-$  is small with respect to  $\mu^+$  in the sense that  $\lambda(\mu) > 1$  then  $(\tilde{\mathbf{L}})$  still holds.

Pinsky [9] treat absolutely continuous potentials  $d\mu = V^+dx - V^-dx$  and prove in [9, Theorem 1.1] that if  $\sup_{x \in \mathbb{R}^d} \mathbf{E}_x(\exp(\int_0^\infty V^-(B_t)dt)) < \infty$ , the Liouville property  $(\tilde{\mathbf{L}})$  is equivalent to

$$\int_0^\infty V^+(B_t)dt = \infty, \mathbf{P}_x\text{-a.e. } (\iff \mathbf{(A)}).$$

We will give an example of potential  $\mu$  that even if  $\sup_{x \in \mathbb{R}^d} \mathbf{E}_x(\exp(A_\infty^\mu)) = \infty$  and  $\mathbf{E}_x(\exp(-A_\infty^\mu)) = 0$ ,  $(\tilde{\mathbf{L}})$  holds (Example 4.1).

## 2. SCHRÖDINGER FORMS

Let  $X$  be a locally compact separable metric space and  $m$  a positive Radon measure on  $X$  with full topological support. Let  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  be a regular Dirichlet form on  $L^2(X; m)$ . We denote by  $u \in \mathcal{D}_{loc}(\mathcal{E})$  if for any relatively compact open set  $D$  there exists a function  $v \in \mathcal{D}(\mathcal{E})$  such that  $u = v$   $m$ -a.e. on  $D$ . We denote by  $\mathcal{D}_e(\mathcal{E})$  the family of  $m$ -measurable functions  $u$  on  $X$  such that  $|u| < \infty$   $m$ -a.e. and there exists an  $\mathcal{E}$ -Cauchy sequence  $\{u_n\}$  of functions in  $\mathcal{D}(\mathcal{E})$  such that  $\lim_{n \rightarrow \infty} u_n = u$   $m$ -a.e. We call  $\mathcal{D}_e(\mathcal{E})$  the *extended Dirichlet space* of  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ .

Let  $\mathbf{M} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \{\mathbf{P}_x\}_{x \in X}, \{X_t\}_{t \geq 0}, \zeta)$  be the symmetric Hunt process generated by  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ , where  $\{\mathcal{F}_t\}_{t \geq 0}$  is the augmented filtration and  $\zeta$  is the lifetime of  $\mathbf{M}$ . Denote by  $\{p_t\}_{t \geq 0}$  and  $\{G_\alpha\}_{\alpha \geq 0}$  the semigroup and resolvent of  $\mathbf{M}$ :

$$p_t f(x) = \mathbf{E}_x(f(X_t)), \quad G_\alpha f(x) = \int_0^\infty e^{-\alpha t} p_t f(x) dt.$$

We assume that  $\mathbf{M}$  satisfies next two conditions:

- **Irreducibility (I).** If a Borel set  $A$  is  $p_t$ -invariant, i.e.,  $p_t(1_A f)(x) = 1_A p_t f(x)$   $m$ -a.e. for any  $f \in L^2(X; m) \cap \mathcal{B}_b(X)$  and  $t > 0$ , then  $A$  satisfies either  $m(A) = 0$  or  $m(X \setminus A) = 0$ . Here  $\mathcal{B}_b(X)$  is the space of bounded Borel functions on  $X$ .
- **Strong Feller Property (SF).** For each  $t$ ,  $p_t(\mathcal{B}_b(X)) \subset C_b(X)$ , where  $C_b(X)$  is the space of bounded continuous functions on  $X$ .

We remark that (SF) implies (AC).

- **Absolute Continuity Condition (AC).** The transition probability of  $\mathbf{M}$  is absolutely continuous with respect to  $m$ ,  $p(t, x, dy) = p(t, x, y)m(dy)$  for each  $t > 0$  and  $x \in X$ .

Under (AC), a non-negative, jointly measurable  $\alpha$ -resolvent kernel  $G_\alpha(x, y)$  exists:

$$G_\alpha f(x) = \int_X G_\alpha(x, y) f(y) m(dy), \quad x \in X, \quad f \in \mathcal{B}_b(X).$$

Moreover,  $G_\alpha(x, y)$  is  $\alpha$ -excessive in  $x$  and in  $y$  ([6, Lemma 4.2.4]). We simply write  $G(x, y)$  for  $G_0(x, y)$ . For a measure  $\mu$ , we define the  $\alpha$ -potential of  $\mu$  by

$$G_\alpha \mu(x) = \int_X G_\alpha(x, y) \mu(dy).$$

**Definition 2.1.** (1) A Dirichlet space  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  on  $L^2(X; m)$  is said to be *transient* if there exists a strictly positive, bounded function  $g$  in  $L^1(X; m)$  such that for  $u \in \mathcal{D}(\mathcal{E})$

$$\int_X |u| g dm \leq \sqrt{\mathcal{E}(u, u)}.$$

- (2) A Dirichlet space  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  on  $L^2(X; m)$  is said to be *recurrent* if the constant function 1 belongs to  $\mathcal{D}_e(\mathcal{E})$  and  $\mathcal{E}(1, 1) = 0$ . Namely, there exists a sequence  $\{u_n\} \subset \mathcal{D}(\mathcal{E})$  such that  $\lim_{n, m \rightarrow \infty} \mathcal{E}(u_n - u_m, u_n - u_m) = 0$  and  $\lim_{n \rightarrow \infty} u_n = 1$   $m$ -a.e.

For other characterizations of transience and recurrence, see [6, Theorem 1.6.2, Theorem 1.6.3].

We define the *(1-)capacity*  $\text{Cap}$  associated with the Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  as follows: for an open set  $O \subset X$ ,

$$\text{Cap}(O) = \inf \{ \mathcal{E}(u, u) + (u, u)_m \mid u \in \mathcal{D}(\mathcal{E}), u \geq 1 \text{ } m\text{-a.e. on } O \}$$

and for a Borel set  $A \subset X$ ,

$$\text{Cap}(A) = \inf \{ \text{Cap}(O) \mid O \text{ is open, } O \supset A \}.$$

A statement depending on  $x \in X$  is said to hold q.e. on  $X$  if there exists a set  $N \subset X$  of zero capacity such that the statement is true for every  $x \in X \setminus N$ . “q.e.” is an abbreviation of “quasi-everywhere”. A real valued function  $u$  defined q.e. on  $X$  is said to be *quasi-continuous* if for any  $\epsilon > 0$  there exists an open set  $G \subset X$  such that  $\text{Cap}(G) < \epsilon$  and  $u|_{X \setminus G}$  is finite and continuous. Here,  $u|_{X \setminus G}$  denotes the restriction of  $u$  to  $X \setminus G$ . Each function  $u$  in  $\mathcal{D}_e(\mathcal{E})$  admits a quasi-continuous version  $\tilde{u}$ , that is,  $u = \tilde{u}$   $m$ -a.e. In the sequel, we always assume that every function  $u \in \mathcal{D}_e(\mathcal{E})$  is represented by its quasi-continuous version.

We call a positive Borel measure  $\mu$  on  $X$  *smooth* if it satisfies the following conditions:

(S1)  $\mu$  charges no set of zero capacity,

(S2) there exists an increasing sequence  $\{F_n\}$  of closed sets that

$$(4) \quad \mu(F_n) < \infty,$$

$$(5) \quad \lim_{n \rightarrow \infty} \text{Cap}(K \setminus F_n) = 0 \quad \text{for any compact set } K.$$

We denote by  $S$  the set of smooth measures.

A stochastic process  $\{A_t\}_{t \geq 0}$  is said to be an *additive functional* (AF in abbreviation) if the following conditions hold:

- (i)  $A_t(\cdot)$  is  $\mathcal{F}_t$ -measurable for all  $t \geq 0$ .
- (ii) There exists a set  $\Lambda \in \mathcal{F}_\infty = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$  such that  $P_x(\Lambda) = 1$ , for q.e.  $x \in X$ ,  $\theta_t \Lambda \subset \Lambda$  for all  $t > 0$ , and for each  $\omega \in \Lambda$ ,  $A_t(\omega)$  is a function satisfying:  $A_0 = 0$ ,  $A_t(\omega) < \infty$  for  $t < \zeta(\omega)$ ,  $A_t(\omega) = A_\zeta(\omega)$  for  $t \geq \zeta$ , and  $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega)$  for  $s, t \geq 0$ .

If an AF  $\{A_t\}_{t \geq 0}$  is positive and continuous with respect to  $t$  for each  $\omega \in \Lambda$ , the AF is called a *positive continuous additive functional* (PCAF in abbreviation). The set of all PCAF's is denoted by  $\mathbf{A}_c^+$ . The family  $S$  and  $\mathbf{A}_c^+$  are in one-to-one correspondence (**Revuz correspondence**) as follows: for each smooth measure  $\mu$ , there exists a unique PCAF  $\{A_t\}_{t \geq 0}$  such that for any  $f \in \mathcal{B}^+(X)$  and  $\gamma$ -excessive function  $h$  ( $\gamma \geq 0$ ), that is,  $e^{-\gamma t} p_t h \leq h$ ,

$$(6) \quad \lim_{t \rightarrow 0} \frac{1}{t} E_{h \cdot m} \left( \int_0^t f(X_s) dA_s \right) = \int_X f(x) h(x) \mu(dx)$$

([6, Theorem 5.1.7]). Here,  $E_{h \cdot m}(\cdot) = \int_X E_x(\cdot) h(x) m(dx)$ . We denote by  $A_t^\mu$  the PCAF corresponding to  $\mu \in S$ . For a signed smooth measure  $\mu = \mu^+ - \mu^-$ , we define  $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ .

We introduce some classes of smooth measures.

**Definition 2.2.** Suppose that  $\mu \in S$  is a positive Radon measure.

- (1) A measure  $\mu$  is said to be in the *Kato class* of  $\mathbf{M}$  ( $\mathcal{K}$  in abbreviation) if

$$\lim_{\alpha \rightarrow \infty} \|G_\alpha \mu\|_\infty = 0.$$

A measure  $\mu$  is said to be in the *local Kato class* ( $\mathcal{K}_{loc}$  in abbreviation) if for any compact set  $K$ ,  $1_K \cdot \mu$  belongs to  $\mathcal{K}$ .

- (2) Suppose that  $\mathbf{M}$  is transient. A measure  $\mu$  is said to be in the class  $\mathcal{K}_\infty$  if for any  $\epsilon > 0$ , there exists a compact set  $K = K(\epsilon)$

$$\sup_{x \in X} \int_{K^c} G(x, y) \mu(dy) < \epsilon.$$

A measure  $\mu$  in  $\mathcal{K}_\infty$  is called *Green-tight*.

We note that every measure treated in this paper is supposed to be Radon. We denote the Green-tight class by  $\mathcal{K}_\infty(G)$  if we would like to emphasize the dependence of the Green kernel. Chen [3] define the Green-tight class in slightly different way; however the two definitions are equivalent under (SF) ([8, Lemma 4.1]).

Let  $\mu = \mu^+ - \mu^- \in \mathcal{K}_{loc} - \mathcal{K}$ . We define the Schrödinger form by

$$(7) \quad \begin{cases} \mathcal{E}^\mu(u, u) = \mathcal{E}(u, u) + \int_X u^2 d\mu \\ \mathcal{D}(\mathcal{E}^\mu) = \mathcal{D}(\mathcal{E}) \cap L^2(X; \mu^+). \end{cases}$$

Denoting by  $\mathcal{L}^\mu = \mathcal{L} - \mu$  the self-adjoint operator generated by the closed symmetric form  $(\mathcal{E}^\mu, \mathcal{D}(\mathcal{E}^\mu))$ ,  $(-\mathcal{L}^\mu u, v)_m = \mathcal{E}^\mu(u, v)$ , we see that the associated semigroup  $\exp(t\mathcal{L}^\mu)$  is expressed as  $\exp(t\mathcal{L}^\mu)f(x) = \mathbf{E}_x(\exp(-A_t^\mu)f(X_t))$  (cf. [1]).

Let  $\mathbf{M}^{\mu^+} = (P_x^{\mu^+}, X_t, \zeta)$  the subprocess of  $\mathbf{M}$  by the multiplicative functional  $\exp(-A_t^{\mu^+})$  and suppose that  $\mathbf{M}^{\mu^+}$  is also strong Feller (For this conditions, refer [4]).

### 3. MAXIMUM PRINCIPLE

In this section we consider the maximum principle for Schrödinger forms. For  $h \in \mathcal{B}(X)$  we denote by  $h^+$  and  $h^-$  the positive and negative part of  $h$ .

**Theorem 3.1.** *Assume (A). Then*

$$\lambda(\mu) > 1 \iff (\mathbf{MP}).$$

*Proof.* For  $h \in \mathcal{H}^{ba}(\mu)$

$$\begin{aligned} h(x) &\leq \mathbf{E}_x(e^{-A_t^\mu} h(X_t)) = \mathbf{E}_x^{\mu^+}(e^{A_t^{\mu^-}} h(X_t)) \leq \mathbf{E}_x^{\mu^+}(e^{A_\zeta^{\mu^-}} h^+(X_t)) \\ &\leq \|h^+\|_\infty \cdot \mathbf{E}_x^{\mu^+}(e^{A_\zeta^{\mu^-}}; t < \zeta). \end{aligned}$$

If  $\lambda(\mu) > 1$ , then  $\sup_{x \in X} \mathbf{E}_x^{\mu^+}(\exp(A_\zeta^{\mu^-})) < \infty$  by [3, Theorem 5.1]. Hence the right-hand side tends to 0 as  $t \rightarrow \infty$  because

$$\lim_{t \rightarrow \infty} \mathbf{P}_x^{\mu^+}(t < \zeta) = \mathbf{E}_x(e^{-A_\infty^{\mu^+}} 1_{\{\zeta = \infty\}}) = 0$$

by Assumption (A).

Suppose  $\lambda(\mu) \leq 1$ . By the definition of  $\lambda(\mu)$

$$(8) \quad \inf \left\{ \mathcal{E}^{\mu^+}(u, u) \mid \lambda(\mu) \int_X u^2 d\mu^- = 1 \right\} = 1.$$

It follows from [12, Lemma 5.16, Corollary 5.17] that the minimizer  $h$  in (8) is a bounded positive continuous with  $p_t^{\mu^+ - \lambda(\mu)\mu^-}$ -invariance,  $h(x) = p_t^{\mu^+ - \lambda(\mu)\mu^-} h(x)$ . Hence

$$h(x) = p_t^{\mu^+ - \lambda(\mu)\mu^-} h(x) \leq p_t^{\mu^+ - \mu^-} h(x) = p_t^\mu h(x),$$

and **(MP)** does not hold.  $\square$

In the sequel of this section, we deal with a strongly local Dirichlet form and extend a result of [2]. We set

$$\begin{aligned} \mathcal{S} &= \left\{ \{x_n\}_{n=1}^\infty \subset X \mid x_n \rightarrow \Delta \text{ and } \lim_{n \rightarrow \infty} \mathbf{E}_{x_n}(e^{-\zeta}) = 1 \right\}, \\ \tilde{\mathcal{S}} &= \left\{ \{x_n\}_{n=1}^\infty \subset X \mid x_n \rightarrow \Delta \text{ and } \lim_{n \rightarrow \infty} \mathbf{P}_{x_n}(\zeta > \epsilon) \rightarrow 0 \text{ for any } \epsilon > 0 \right\}. \end{aligned}$$

**Lemma 3.1.** *It holds that*

$$\mathcal{S} = \tilde{\mathcal{S}}.$$

*Proof.* For  $\{x_n\}_{n=1}^\infty \in \mathcal{S}$

$$\mathbf{E}_{x_n}(e^{-\zeta}) \leq e^{-\epsilon} \mathbf{P}_{x_n}(\zeta > \epsilon) + \mathbf{P}_{x_n}(\zeta \leq \epsilon) = 1 - (1 - e^{-\epsilon}) \mathbf{P}_{x_n}(\zeta > \epsilon),$$

and thus

$$\overline{\lim}_{n \rightarrow \infty} \mathbf{P}_{x_n}(\zeta > \epsilon) \leq \overline{\lim}_{n \rightarrow \infty} \frac{1 - \mathbf{E}_{x_n}(e^{-\zeta})}{1 - e^{-\epsilon}} = 0.$$

For  $\{x_n\}_{n=1}^\infty \in \tilde{\mathcal{S}}$

$$\mathbf{E}_{x_n}(e^{-\zeta}) = \mathbf{E}_{x_n}(e^{-\zeta}; \zeta > \epsilon) + \mathbf{E}_{x_n}(e^{-\zeta}; \zeta \leq \epsilon) \geq e^{-\epsilon} \mathbf{P}_{x_n}(\zeta \leq \epsilon),$$

and thus  $\underline{\lim}_{n \rightarrow \infty} \mathbf{E}_{x_n}(e^{-\zeta}) \geq e^{-\epsilon}$  and  $\lim_{n \rightarrow \infty} \mathbf{E}_{x_n}(e^{-\zeta}) = 1$ .  $\square$

A Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is said to be *strongly local*, if  $\mathcal{E}(u, v) = 0$  for any  $u, v \in \mathcal{D}(\mathcal{E})$  such that  $u$  is constant on a neighborhood of  $\text{supp}[v]$ . In the sequel of this section, we assume that  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local. We introduce

$$\tilde{\mathcal{H}}^{ba}(\mu) = \left\{ h \mid \begin{array}{l} h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C(X) \text{ is bounded above, } \mathcal{E}^\mu(h, \varphi) \leq 0 \text{ for any} \\ \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0^+(X), \overline{\lim}_{n \rightarrow \infty} h(x_n) \leq 0 \text{ for any } \{x_n\}_{n=1}^\infty \in \mathcal{S}. \end{array} \right\}.$$

**Lemma 3.2.** *Let  $\{\tau_n\}_{n=1}^\infty$  be a sequence of stopping times such that  $\tau_n < \zeta$  and  $\tau_n \uparrow \zeta$ , as  $n \rightarrow \infty$ ,  $\mathbf{P}_x$ -a.s. Then there exists a subsequence  $\{\sigma_n\}_{n=1}^\infty$  of  $\{\tau_n\}_{n=1}^\infty$  such that*

$$(9) \quad \mathbf{P}_x(\{X_{\sigma_n}\} \in \mathcal{S}) = 1.$$

*Proof.* First note

$$\begin{aligned} \{\zeta(\theta_{\tau_n}) > \epsilon, \tau_n < \zeta\} &= \{\tau_n + \zeta(\theta_{\tau_n}) > \tau_n + \epsilon, \tau_n < \zeta\} \\ &= \{\zeta > \tau_n + \epsilon, \tau_n < \zeta\} = \{\zeta > \tau_n + \epsilon\}. \end{aligned}$$

We then have by the strong Markov property

$$\begin{aligned}\mathbf{E}_x(\mathbf{P}_{X_{\tau_n}}(\zeta > \epsilon)) &= \mathbf{E}_x(\mathbf{P}_{X_{\tau_n}}(\zeta > \epsilon); \tau_n < \zeta) \\ &= \mathbf{E}_x(\mathbf{P}_x(\zeta(\theta_{\tau_n}) > \epsilon, \tau_n < \zeta | \mathcal{F}_{\tau_n})) \\ &= \mathbf{P}_x(\zeta > \tau_n + \epsilon) \longrightarrow 0 \text{ as } n \rightarrow \infty.\end{aligned}$$

Hence there exists a subsequence  $\{\tau_n^{(1)}\}_{n=1}^\infty$  of  $\{\tau_n\}_{n=1}^\infty$  such that

$$\mathbf{P}_{X_{\tau_n^{(1)}}}(\zeta > 1) \longrightarrow 0 \text{ as } n \rightarrow \infty, \text{ } \mathbf{P}_x\text{-a.s..}$$

By the same argument

$$\mathbf{E}_x(\mathbf{P}_{X_{\tau_n^{(1)}}}(\zeta > 1/2); \tau_n^{(1)} < \zeta) \longrightarrow 0$$

and there exists a subsequence  $\{\tau_n^{(2)}\}_{n=1}^\infty$  of  $\{\tau_n^{(1)}\}_{n=1}^\infty$  such that

$$\mathbf{P}_{X_{\tau_n^{(2)}}}(\zeta > 1/2) \longrightarrow 0 \text{ as } n \rightarrow \infty, \text{ } \mathbf{P}_x\text{-a.s.}$$

By continuing this procedure we can take a subsequence  $\{\tau_n^{(k)}\}_{n=1}^\infty$  of  $\{\tau_n^{(k-1)}\}_{n=1}^\infty$  such that

$$\mathbf{P}_{X_{\tau_n^{(k)}}}(\zeta > 1/k) \longrightarrow 0 \text{ as } n \rightarrow \infty, \text{ } \mathbf{P}_x\text{-a.s.}$$

The sequence  $\{\sigma_n := \tau_n^{(n)}\}_{n=1}^\infty$  is a desired one.  $\square$

**Lemma 3.3.** *Suppose  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local. Let  $\{D_n\}_{n=1}^\infty$  be a sequence of relatively compact open sets such that  $D_n \uparrow X$ . Define  $S_n = \inf\{t > 0 \mid A_t^\mu > n\}$  and  $T_n = S_n \wedge \tau_{D_n}$ . Then for  $h \in \hat{\mathcal{H}}^{ba}(\mu)$*

$$\mathbf{E}_x \left( e^{-A_{T_n \wedge t}^\mu} h(X_{T_n \wedge t}) \right) \geq h(x) \text{ q.e. } x.$$

*Proof.* This lemma can be derived by the argument similar to that in [12, Lemma 4.7]. In fact, put  $\mathcal{L} = \mathcal{D}(\mathcal{E}) \cap C_0(X)$ . Then  $\mathcal{L}$  is a *Stone vector lattice*, i.e., if  $f, g \in \mathcal{L}$ , then  $f \vee g \in \mathcal{L}$ ,  $f \wedge 1 \in \mathcal{L}$ . For  $h \in \hat{\mathcal{H}}^{ba}(\mu)$  define the functional  $I$  by

$$(10) \quad I(\varphi) = -\mathcal{E}^\mu(h, \varphi), \quad \varphi \in \mathcal{L}.$$

Then  $I(\varphi)$  is a pre-integral, that is,  $I(\varphi_n) \downarrow 0$  whenever  $\varphi_n \in \mathcal{L}$  and  $\varphi_n(x) \downarrow 0$  for all  $x \in X$ . Indeed, let  $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$  such that  $\psi = 1$  on  $\text{supp}[\varphi_1]$ . Then  $\varphi_n \leq \|\varphi_n\|_\infty \psi$  and

$$I(\varphi_n) \leq \|\varphi_n\|_\infty \cdot I(\psi) \downarrow 0, \quad n \rightarrow \infty.$$

Notice that by the regularity of  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  the smallest  $\sigma$ -field generated by  $\mathcal{L}$  is identical with the Borel  $\sigma$ -field. We then see from [5, Theorem 4.5.2] that there exists a positive Borel measure  $\nu$  such that

$$(11) \quad I(\varphi) = \int_X \varphi d\nu.$$

By the definition of  $\nu$  we see that  $\nu$  is a Radon measure and satisfies (S2) for any increasing sequence  $\{F_n\}$  of compact sets with  $F_n \uparrow X$ . Let  $K$  be a compact set of zero capacity. Then for a relatively compact open set  $D$  such that  $K \subset D$ , there exists a sequence  $\{\varphi_n\} \subset \mathcal{D}(\mathcal{E}) \cap C_0^+(D)$  such that  $\varphi_n \geq 1$  on  $K$  and  $\mathcal{E}_1(\varphi_n, \varphi_n) \rightarrow 0$  as  $n \rightarrow \infty$  ([6, Lemma 2.2.7]). For  $\psi \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$  with  $\psi = 1$  on  $D$ ,

$$I(\varphi_n) = -\mathcal{E}^\mu(h, \varphi_n) = -\mathcal{E}^\mu(h\psi, \varphi_n) \leq \mathcal{E}^{|\mu|}(h\psi, h\psi)^{1/2} \cdot \mathcal{E}^{|\mu|}(\varphi_n, \varphi_n)^{1/2},$$



where  $|\mu| = \mu^+ + \mu^-$ . Note that  $1_D|\mu| \in \mathcal{K}$  and  $\|G_1(1_D|\mu|)\|_\infty < \infty$ . We then see from the Stollmann-Voigt inequality ([10]) that

$$\int_X \varphi_n^2 d|\mu| = \int_X \varphi_n^2 1_D d|\mu| \leq \|G_1(1_D|\mu|)\|_\infty \cdot \mathcal{E}_1(\varphi_n, \varphi_n) \longrightarrow 0, \quad n \rightarrow \infty$$

and  $\mathcal{E}^{|\mu|}(\varphi_n, \varphi_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Since

$$\nu(K) \leq \int_X \varphi_n d\nu = I(\varphi_n) \rightarrow 0, \quad n \rightarrow \infty$$

$\nu$  satisfies (S1), consequently the measure  $\nu$  is smooth.

The equations (10), (11) lead us to

$$\mathcal{E}(h, \varphi) = - \int_X \varphi h d\mu - \int_X \varphi d\nu = - \int_X \varphi (h d\mu + d\nu).$$

On account of [6, Theorem 5.5.5], we have

$$h(X_t) = h(X_0) + M_t^{[h]} + \int_0^t h(X_s) dA_s^\mu + A_t^\nu \quad \mathbf{P}_x\text{-a.s., q.e. } x.$$

Hence, by Itô's formula

$$\begin{aligned} e^{-A_t^\mu} h(X_t) &= h(X_0) + \int_0^t e^{-A_s^\mu} dM_s^{[h]} + \int_0^t e^{-A_s^\mu} h(X_s) dA_s^\mu \\ &\quad + \int_0^t e^{-A_s^\mu} dA_s^\nu - \int_0^t e^{-A_s^\mu} h(X_s) dA_s^\mu \\ &= h(X_0) + \int_0^t e^{-A_s^\mu} dM_s^{[h]} + \int_0^t e^{-A_s^\mu} dA_s^\nu \quad \mathbf{P}_x\text{-a.s., q.e. } x. \end{aligned}$$

Since  $\int_0^{T_n \wedge t} e^{-A_s^\mu} dM_s^{[h]}$  is a martingale and  $\int_0^t e^{-A_s^\mu} dA_s^\nu \geq 0$ ,

$$E_x \left( e^{-A_{T_n \wedge t}^\mu} h(X_{T_n \wedge t}) \right) \geq h(x) \quad \text{q.e. } x.$$

□

**Lemma 3.4.** *It holds that*

$$\tilde{\mathcal{H}}^{ba}(\mu) \subset \mathcal{H}^{ba}(\mu).$$

*Proof.* Let  $h$  be a function in  $\tilde{\mathcal{H}}^{ba}(\mu)$  and  $\{T_n\}_{n=1}^\infty$  a sequence of stopping times defined in Lemma 3.3. We fix a point  $x \in X$  such that

$$\mathbf{E}_x \left( e^{-A_{T_n \wedge t}^\mu} h(X_{T_n \wedge t}) \right) \geq h(x).$$

Since  $T_n < \zeta$  and  $T_n \uparrow \zeta$ , we can take a subsequence  $\{\sigma_n\}$  of  $\{T_n\}$  satisfying (9) in Lemma 3.2. Since  $h^+$  is bounded continuous and

$$\overline{\lim}_{n \rightarrow \infty} h^+(X_{\sigma_n \wedge t}) = 0, \quad \mathbf{P}_x\text{-a.s. on } \{t \geq \zeta\}$$

by (9), we have

$$\begin{aligned}
& \overline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}) \right) \\
& \leq \overline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}); t < \zeta \right) + \overline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}); t \geq \zeta \right) \\
& \leq \mathbf{E}_x \left( \left( e^{-A_t^\mu} h^+(X_t); t < \zeta \right) + \mathbf{E}_x \left( \overline{\lim}_{n \rightarrow \infty} e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}); t \geq \zeta \right) \right) \\
& = \mathbf{E}_x \left( e^{-A_t^\mu} h^+(X_t) \right).
\end{aligned}$$

Here, the second inequality above follows from the inverse Fatou's lemma because

$$e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}) \leq e^{A_{\sigma_n \wedge t}^{\mu^-}} h^+(X_{\sigma_n \wedge t}) \leq \|h^+\|_\infty \cdot e^{A_t^{\mu^-}} \in L^1(\mathbf{P}_x)$$

by  $\mu^- \in \mathcal{K}$ .

Besides, we have

$$\begin{aligned}
& \underline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^-(X_{\sigma_n \wedge t}) \right) \\
& \geq \underline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^-(X_{\sigma_n \wedge t}); t < \zeta \right) + \underline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^-(X_{\sigma_n \wedge t}); t \geq \zeta \right) \\
& \geq \mathbf{E}_x \left( \underline{\lim}_{n \rightarrow \infty} e^{-A_{\sigma_n \wedge t}^\mu} h^-(X_{\sigma_n \wedge t}); t < \zeta \right) = \mathbf{E}_x \left( e^{-A_t^\mu} h^-(X_t) \right).
\end{aligned}$$

Hence

$$\begin{aligned}
h(x) & \leq \underline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h(X_{\sigma_n \wedge t}) \right) \\
& \leq \overline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^+(X_{\sigma_n \wedge t}) \right) - \underline{\lim}_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{\sigma_n \wedge t}^\mu} h^-(X_{\sigma_n \wedge t}) \right) \\
& \leq \mathbf{E}_x \left( e^{-A_t^\mu} h^+(X_t) \right) - \mathbf{E}_x \left( e^{-A_t^\mu} h^-(X_t) \right) = \mathbf{E}_x \left( e^{-A_t^\mu} h(X_t) \right),
\end{aligned}$$

and  $h(x) \leq p_t^\mu h(x)$  for q.e.  $x$ . Since  $p_t^\mu$  is strong Feller,  $p_t^\mu(h \vee (-n))(x) \geq h(x)$  for all  $x \in X$  and  $p_t^\mu h(x) \geq h(x)$  for all  $x \in X$  by letting  $n \rightarrow \infty$ .  $\square$

Following [2], we define the *refined maximum principle*:

**(RMP)** If  $h \in \tilde{\mathcal{H}}^{ba}(\mu)$ , then  $h(x) \leq 0$  for all  $x \in X$ .

Combining Lemma 3.4 with Theorem 3.1, we have the next theorem.

**Theorem 3.2.** *Suppose  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local. Then under Assumption (A)*

$$\lambda(\mu) > 1 \implies \textbf{(RMP)}.$$

**Remark 3.1.** Suppose  $D$  is a bounded domain in  $\mathbb{R}^d$  and consider the absorbing Brownian motion  $(\mathbf{P}_x, B_t, \tau_D)$  on  $D$ , where  $\tau_D$  is the first exit time from  $D$ . If  $D$  is Green-bounded, i.e.,  $\sup_{x \in D} \mathbf{E}_x(\tau_D) < \infty$ , then  $\mathcal{S}$  is identical to the set of sequences  $\{x_n\}$  such that  $x_n \rightarrow \partial D$  and  $E_{x_n}(\tau_D) \rightarrow 0$  as  $n \rightarrow \infty$ . Indeed, take  $\delta > 0$  so that  $\sup_{x \in D} \mathbf{E}_x(\delta \tau_D) < 1$ . Then since  $\sup_{x \in D} \mathbf{E}_x(\exp(\delta \tau_D)) < \infty$  by Has'minskii's lemma, we see  $\sup_{x \in D} \mathbf{E}_x(\tau_D^2) < \infty$ . Hence if  $\mathbf{P}_{x_n}(\tau_D > \epsilon) \rightarrow 0$  as  $n \rightarrow \infty$ , then

$$E_{x_n}(\tau_D) \leq (E_{x_n}(\tau_D^2))^{1/2} \cdot (\mathbf{P}_{x_n}(\tau_D > \epsilon))^{1/2} + \epsilon \mathbf{P}_{x_n}(\tau_D \leq \epsilon) \rightarrow \epsilon.$$

Since  $\epsilon \mathbf{P}_{x_n}(\tau_D > \epsilon) \leq E_{x_n}(\tau_D)$ , the converse follows from Lemma 3.1.

Let

$$\lambda_0 = \inf \left\{ \frac{1}{2} \mathbb{D}(v, v) + \int_D v^2 d\mu \mid v \in H_0^1(D), \int_D v^2 dx = 1 \right\},$$

where  $\mathbb{D}$  is the classical Dirichlet integral. We see from [2, Theorem 1.1] that

$$\lambda_0 > 0 \iff (\mathbf{RMP}).$$

Moreover, we see from Lemma 3.5 below that if  $D$  is bounded, then  $\lambda_0 > 0$  and  $\lambda(\mu) > 1$  are equivalent, and so

$$\lambda(\mu) > 1 \iff (\mathbf{RMP}).$$

We remark that  $\lambda_0 > 0$  implies  $\lambda(\mu) > 1$  for a general domain  $D$  (Lemma 3.5 below), while  $\lambda(\mu) > 1$  does not imply  $\lambda_0 > 0$  in general. In fact, consider  $\mathcal{L}u = (1/2)u'' - \mu u$  ( $\mu = \alpha\delta_{-1} - \beta\delta_1$ ,  $\alpha > 0$ ,  $\beta > 0$ ) on  $\mathbb{R}^1$ . We define

$$\lambda(\alpha, \beta) := \lambda(\mu) = \inf \left\{ \frac{1}{2} \mathbb{D}(u, u) + \alpha u(-1)^2 \mid u \in H^1(\mathbb{R}^1), \beta u(1)^2 = 1 \right\}$$

and

$$\lambda_0(\alpha, \beta) := \inf \left\{ \frac{1}{2} \mathbb{D}(u, u) + \alpha u(-1)^2 - \beta u(1)^2 \mid u \in H^1(\mathbb{R}^1), \int_{\mathbb{R}^1} u^2 dx = 1 \right\}.$$

Denote by  $\mathcal{L}_0$  the operator  $1/2(d^2/dx^2) - \alpha\delta_{-1}$ . By the Dirichlet principle, the infimum of  $\lambda(\alpha, \beta)$  is attained by the  $\mathcal{L}_0$ -harmonic function  $u_0$  with  $u_0(1) = 1/\sqrt{\beta}$ , i.e.,

$$u_0(x) = \begin{cases} \gamma, & x \leq -1, \\ \gamma + \frac{1/\sqrt{\beta} - \gamma}{2}(x+1), & -1 \leq x < 1, \\ 1/\sqrt{\beta}, & x \geq 1. \end{cases}$$

Here,  $\gamma$  is determined by

$$\mathcal{L}_0 u_0(-1) = 0 \iff \frac{u_0'(-1+) - u_0'(-1-)}{2} = \alpha u_0(-1) \iff \frac{1/\sqrt{\beta} - \gamma}{4} = \alpha \gamma,$$

and thus  $\gamma = 1/(\sqrt{\beta}(4\alpha + 1))$ . Note that  $u_0$  belongs to the extended Dirichlet space  $H_e^1(\mathbb{R}^1) (\supset H^1(\mathbb{R}^1))$  (cf. [6, Exercise 6.4.9]). We then see that

$$\lambda(\alpha, \beta) = \frac{1}{2} \int_{-1}^1 \left( \frac{du_0}{dx} \right)^2 dx + \alpha u_0(-1)^2 = \frac{\alpha}{\beta(4\alpha + 1)}.$$

For  $\beta < 1/4$ , let  $\alpha_0 = \beta/(1 - 4\beta)$ . Then  $\lambda(\alpha_0, \beta) = 1$  and  $\lambda(\alpha, \beta) > 1$  for  $\alpha > \alpha_0$ . We see from [13, Lemma 2.2] that  $\lambda(\alpha, \beta) \geq 1$  is equivalent with  $\lambda_0(\alpha, \beta) \geq 0$ . Noting that  $\lambda_0(\alpha, \beta) \leq 0$  for any  $\alpha, \beta$ , we see that for  $\beta < 1/4$  and  $\alpha > \beta/(1 - 4\beta)$ ,  $\lambda_0(\alpha, \beta) = 0$  and  $\lambda(\alpha, \beta) > 1$ .

**Lemma 3.5.** *It holds that*

$$\lambda_0 := \inf \left\{ \mathcal{E}^\mu(u, u) \mid u \in \mathcal{D}(\mathcal{E}), \int_X u^2 dm = 1 \right\} > 0 \implies \lambda(\mu) > 1.$$

*If there exists a positive constant  $C$  such that*

$$\int_X u^2 dm \leq C \mathcal{E}^{\mu^+}(u, u),$$

*then the converse also holds.*

*Proof.* Let  $\varphi_0 \in \mathcal{D}_\varepsilon(\mathcal{E}^{\mu^+})$  be the minimizer in (1):

$$\lambda(\mu) = \mathcal{E}^{\mu^+}(\varphi_0, \varphi_0), \quad \int_X \varphi_0^2 d\mu^- = 1.$$

If  $\lambda_0 > 0$ , then

$$\lambda(\mu) - 1 = \mathcal{E}^{\mu^+}(\varphi_0, \varphi_0) - \int_X \varphi_0^2 d\mu^- = \mathcal{E}^\mu(\varphi_0, \varphi_0) = \lambda_0 \int_X \varphi_0^2 dm > 0.$$

If  $\lambda(\mu) > 1$ , then for any  $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}^{\mu^+}(u, u) - \lambda(\mu) \int_X u^2 d\mu^- \geq 0 \iff \lambda(\mu) \cdot \mathcal{E}^\mu(u, u) \geq (\lambda(\mu) - 1) \cdot \mathcal{E}^{\mu^+}(u, u).$$

Hence by the assumption,

$$\mathcal{E}^\mu(u, u) \geq \frac{(\lambda(\mu) - 1)}{C\lambda(\mu)} \int_X u^2 dm.$$

□

#### 4. LIOUVILLE PROPERTY

Let us introduce the set of  $p_t^\mu$ -invariant bounded functions by

$$\mathcal{H}^b(\mu) = \{h \in \mathcal{B}_b(X) \mid p_t^\mu h = h\}.$$

We here define the Liouville property **(L)** by

**(L)** If  $h \in \mathcal{H}^b(\mu)$ , then  $h(x) = 0$  for all  $x \in X$ .

**Corollary 4.1.** *Suppose **(A)**. Then*

$$\lambda(\mu) > 1 \implies \textbf{(L)}.$$

*Proof.* Let

$$\mathcal{H}^{bb}(\mu) = \{h \in \mathcal{B}(X) \mid h \text{ is bounded below, } p_t^\mu h \leq h\}.$$

We see, by the same argument as in Theorem 3.1, that an element  $h$  in  $\mathcal{H}^{bb}(\mu)$  satisfies  $h(x) \geq 0$  for any  $x \in X$ . Since  $\mathcal{H}^b(\mu) = \mathcal{H}^{ba}(\mu) \cap \mathcal{H}^{bb}(\mu)$ , this corollary is derived. □

For a strongly local Dirichlet form  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  we set

$$\tilde{\mathcal{H}}^b(\mu) = \{h \in \mathcal{D}_{loc}(\mathcal{E}) \cap C_b(X) \mid \mathcal{E}^\mu(h, \varphi) = 0, \forall \varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(X)\}.$$

**Lemma 4.1.** *Assume  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local. If  $\mathbf{M}$  is conservative, then  $\tilde{\mathcal{H}}^b(\mu) \subset \mathcal{H}^b(\mu)$ .*

*Proof.* For  $h \in \tilde{\mathcal{H}}^b(\mu)$  let  $\{T_n\}_{n=1}^\infty$  be the sequence of stopping times defined in Lemma 3.3. Then  $\mathbf{E}_x(\exp(-A_{T_n \wedge t}^\mu)h(X_{T_n \wedge t})) = h(x)$  for any  $n$ . Noticing that  $T_n \rightarrow \infty$ ,  $\mathbf{P}_x$ -a.s. by the conservativeness of  $\mathbf{M}$  and that  $\exp(-A_{T_n \wedge t}^\mu)h(X_{T_n \wedge t}) \leq \|h\|_\infty \exp(A_t^{\mu^-}) \in L^1(\mathbf{P}_x)$ , we have

$$h(x) = \lim_{n \rightarrow \infty} \mathbf{E}_x \left( e^{-A_{T_n \wedge t}^\mu} h(X_{T_n \wedge t}) \right) = \mathbf{E}_x \left( e^{-A_t^\mu} h(X_t) \right)$$

by the dominated convergence theorem. □

Define the property  $(\tilde{\mathbf{L}})$  by

$(\tilde{\mathbf{L}})$  If  $h \in \tilde{\mathcal{H}}^b(\mu)$ , then  $h(x) = 0$  for all  $x \in X$ .

Lemma 4.1 leads us to the next corollary.

**Corollary 4.2.** *Suppose  $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$  is strongly local and  $\mathbf{M}$  is conservative. Then under Assumption  $(\mathbf{A})$*

$$\lambda(\mu) > 1 \implies (\tilde{\mathbf{L}}).$$

We finally give a Schrödinger operator,  $-1/2\Delta + \mu$  which satisfies  $(\tilde{\mathbf{L}})$ ; however, the positive part and negative part of potential  $\mu$  satisfy

$$\mathbf{E}_x \left( e^{-A_\infty^{\mu^+}} \right) = 0, \quad \sup_{x \in \mathbb{R}^d} \mathbf{E}_x \left( e^{A_\infty^{\mu^-}} \right) = \infty.$$

**Example 4.1.** Let us define

$$\lambda_1 = \inf \left\{ \frac{1}{2} \mathbb{D}(u, u) \mid u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 d\sigma = 1 \right\}$$

and

$$\lambda_2 = \inf \left\{ \frac{1}{2} \mathbb{D}(u, u) + (u, u)_m \mid u \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 d\sigma = 1 \right\},$$

where  $m$  is the Lebesgue measure and  $\sigma$  the measure such that  $\sigma|_{\partial B(0,1)}$  is the surface measure of  $\partial B(0,1)$  and  $\sigma(\mathbb{R}^d \setminus \partial B(0,1)) = 0$ . Let  $\mu = m - \gamma\sigma$ , that is,  $\mu^+ = m$ ,  $\mu^- = \gamma\sigma$  ( $\gamma > 0$ ). Note that  $A_t^m = t$  and  $A_t^\sigma$  is the local time of the unit sphere. We see that if  $\lambda_1 < \gamma < \lambda_2$ , then  $\lambda(\mu) > 1$ , and  $-1/2\Delta + \mu$  satisfies  $(\tilde{\mathbf{L}})$ ; however,  $\mathbf{E}_x(\exp(A_\infty^\sigma)) = \infty$ .

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